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A new method to study nonstationary systems and channels that applies to both the time-variant and time-invariant case has been developed. It is shown that a significant simplification occurs when systems are formulated in phase space. The phase space can be time-frequency for the usual formulation of channels/systems, or position-wave number for dispersive channels and pulses. In the standard textbook approach of input-output relations and system functions, the relations are formulated in time or frequency. We have derived input-output relations for time-varying channels. The method has led to practical solutions to a number of problems, both deterministic and random. Among them are defining the concept of quasi-stationarity, formulating features that are dispersion invariant, obtaining simple characterizations of pulses that propagate in dispersive channels with attenuation. In addition, a simple and accurate approximate phase space method is given that allows the calculation of features easily. The method is also applied to the time evolution and propagation of noise fields. It is shown how differential equations can be transformed into phase space, and it is shown how that can be done for equations governing pulse propagation in dispersive channels.

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A New Approach to the Characterization and Detection of Nonstationary Environments and Channels

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Contents

1 Introduction

We have developed a new method to study nonstationary signals and systems. Modern engineering has developed the concept and methods generally called "system theory" or "input-output relations" and the associated concepts. This has become textbook material. We believe we generalized the concept of "input-output relations" by developing a new method to study systems and channels that applies to both the time-variant and time-invariant case. We have shown that a remarkable simplification occurs, both conceptually and technically, when systems are formulated in phase space. The phase space can be time-frequency for the usual formulation of channels/systems, but can also be position-wave number, for systems such as pulse propagation and array processing. Our method has led to practical solutions to many problems, both deterministic and random. The fundamental idea of our approach can be best conveyed by a contrast with the traditional method. The standard textbook approach which has been the foundation of input-output relations is the concept of a "system function", or Green's function, or impulse response function. If we have an input time function, $f(t)$, which passes through a system characterized by a system function $h(t)$, then the output is given by $x(t)$ and is commonly symbolized as in Fig. 1.

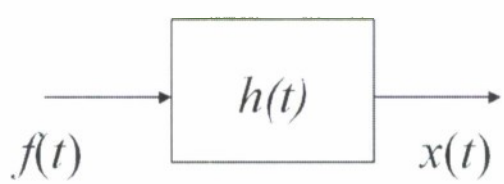

Figure 1: Input-output representation of a system in the time domain.

For many reasons, one often transforms to the frequency domain (Fourier domain). Historically the most important reason has been that in that domain the relationship between input, output, and system function is simple for time-invariant systems. In addition, and perhaps more importantly, the reason for going into the Fourier domain is that one can gain insight into the nature of the solution. If the input, system and output transforms of the time functions are given by $F(\omega)$, $H(\omega)$ and $X(\omega)$ respectively, then the input-output relations are symbolized as in Fig. 2.

However, we have argued and have shown that this standard formulation does not fully describe what is happening if the system or signals are nonstationary, and that a significant simplification occurs if we have input-output relations in the *time-frequency* plane, as symbolized in Fig. 3. where $C_f(t,\omega)$ and $C_x(t,\omega)$ are the input and output time-frequency distributions. The box with the question mark is meant to symbolize the time-frequency system function; we believe we have

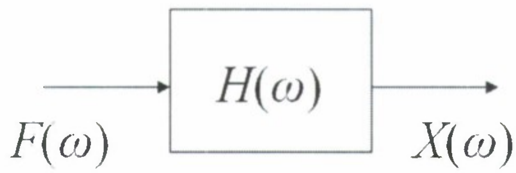

Figure 2: Input-output representation of a system in the frequency domain.

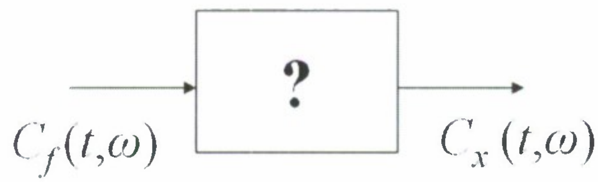

Figure 3: Input-output representation of a system in the time-frequency domain.

solved the system approach as exemplified by the question mark, and have solved a number of problems to show the effectiveness of the method. In addition we point out that this approach lends itself to powerful approximation methods that give considerable insight into the nature of time-variant systems.

Transformation of input-output relations into phase space. We write the input-output relations in the form

$$L[t]x(t) = f(t) \tag{1}$$

where L is a linear (possible time-dependent channel). The system function can be a random function as we will describe consequently. For the governing system we take the linear time-variant system

$$a_n(t)\frac{d^n x(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x(t)}{dt^{n-1}} \cdots + a_1(t)\frac{dx(t)}{dt} + a_0(t)x(t) = f(t) \tag{2}$$

where

$$L(D,t) = a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} \cdots + a_1(t)\frac{d}{dt} + a_0(t) \tag{3}$$

$$D = \frac{d}{dt} \tag{4}$$

For the moment, for the time-frequency distribution we take the Wigner distribution defined by

$$W_{x,x}(t,\omega) = \frac{1}{2\pi}\int x^*(t-\tfrac{1}{2}\tau)\,x(t+\tfrac{1}{2}\tau)\,e^{-i\tau\omega}\,d\tau \tag{5}$$

We now state our main result: We write Eq. (2) in the following form

$$L(D,t)x(t) = f(t) \tag{6}$$

then the input output time-frequency relations are

$$L^*(A,\mathcal{E})L(B,\mathcal{F})W_{x,x}(t,\omega) = W_{f,f}(t,\omega) \tag{7}$$

where

$$A = \frac{1}{2}\frac{\partial}{\partial t} - j\omega \qquad B = \frac{1}{2}\frac{\partial}{\partial t} + j\omega \tag{8}$$

$$\mathcal{E} = \frac{1}{2i}\frac{\partial}{\partial\omega} + t \qquad \mathcal{F} = -\frac{1}{2i}\frac{\partial}{\partial\omega} + t \tag{9}$$

The above are the basis of our approach, however, we emphasize that we have done this in more general ways by developing input-output relations when partial differential equations are the governing equations, and also we have considered the random case. We now describe some of the main results we have obtained. For ease of readership we have tried to make each section independent of each other and have, hence, repeated some of the equations rather then referring the reader to other sections.

2 Results

2.1 Quasi-Stationary channels

Going from the time-invariant channels case to the time-variant case is a big jump, and one naturally may consider the in between case, which we call "quasi stationary" or "locally stationary". By "locally stationary" we mean a system that is not stationary, but which is close to stationary for an *interval* of time. We have derived a criterion for local stationary in terms of time-frequency concepts. The basis of our derivation is simple and physical, namely that if around a time point the properties of the distribution does not change for that particular signal, then we have a situation that is indeed stationary around that time. We have obtained the following criterion: we consider a time point t_0 and *if*

$$\overline{W}(t_0,\omega) \sim \overline{W}_t(t_0)\overline{W}_\omega(\omega) \text{ in the neighborhood of } t_0 \tag{10}$$

then we say that the process is locally stationary around the time point t_0.

Also, if in an interval around a particular frequency point ω_0

$$W(t,\omega_0) \sim W_t(t)W_\omega(\omega_0) \text{ in the neighborhood of } \omega_0 \tag{11}$$

then we will say that the process is locally stationary in the frequency variable. One can also define local stationarity in a time-frequency region. If at a time frequency point t_0, ω_0 we have that

$$W(t_0, \omega_0) \sim W_t(t_0) W_\omega(\omega_0) \text{ in the neighborhood of} t_0,\omega_0 \tag{12}$$

then one would say that the process is stationary in a region. For this to be so, both Eq. (10) and (11) must hold. In addition we have defined a measure of local stationarity by how much the joint distribution deviates from a product form. We use

$$\epsilon^2 = \int \int_{t_0-\delta}^{t_0+\delta} \left(\overline{W}(t_0, \omega) - \overline{W}(t_0) \overline{W}(\omega) \right)^2 dt d\omega \tag{13}$$

and similar criteria can be written for other types of time-frequency distribution. We have studied model systems and have found the above criterion to be effective.

2.2 Other Joint Representations

While we have used the Wigner representation, it is important to also do it for other representations, in order to ascertain which distributions are appropriate for different situations. We have obtained results for a number of special cases that are of particular interest.

Ambiguity function domain. The ambiguity function is defined by

$$\mathcal{A}_x(t, \omega) = \frac{1}{2\pi} \int x^*(t - \tau/2) x(t + \tau/2) e^{i\theta t} dt \tag{14}$$

and has been the main tool for radar and sonar systems. We have shown that

$$L^*(A_a, \mathcal{E}_a) L(B_a, \mathcal{F}_a) \mathcal{A}_x(t, \omega) = \mathcal{A}_f(t, \omega) \tag{15}$$

where

$$A_a = -\frac{1}{2} i\theta - \frac{\partial}{\partial \tau}, \qquad B_a = -\frac{1}{2} i\theta + \frac{\partial}{\partial \tau} \tag{16}$$

$$\mathcal{E}_a = \frac{1}{i} \frac{\partial}{\partial \theta} - \frac{1}{2}\tau, \qquad \mathcal{F}_a = \frac{1}{i} \frac{\partial}{\partial \theta} + \frac{1}{2}\tau \tag{17}$$

Two time auto correlation function equation. We have obtained the equation for the two-time autocorrelation function. The two-time autocorrelation is the simplest quantity that characterizes how a random process $x(t)$ is correlated with itself at two different times t_1, t_2. It is defined by

$$R_x(t_1, t_2) = E[x(t_1) x^*(t_2)] \tag{18}$$

where $E[\ ]$ signifies ensemble averaging. For the differential equations given by Eq. (2), where now $x(t)$ is a random variable and $f(t)$ is a random driving process, the autocorrelation function $R_x(t_1, t_2)$ satisfies

$$L\left(\frac{\partial}{\partial t_1}, t_1\right) L^*\left(\frac{\partial}{\partial t_2}, t_2\right) R_x(t_1, t_2) = R_f(t_1, t_2) \tag{19}$$

Continuous Wavelet Transform (CWT). The CWT of a signal $x(t)$ is defined as

$$C_x(a,b) = \frac{1}{\sqrt{a}} \int \psi^* \left(\frac{t-b}{a}\right) x(t) dt \tag{20}$$

where $a > 0$, $-\infty < b < +\infty$. The equation for the CWT is

$$L(A_w) C_x(a,b) = C_f(a,b) \tag{21}$$

where

$$A_w = \frac{\partial}{\partial b} \tag{22}$$

2.3 Variance of a time-variant process

In studying time-variant channels it is important to understand by how much they are varying as a function of time. We have shown that the exact instantaneous variance of a nonstationary random process can be obtained from the Wigner spectrum representation. What is advantageous in the evaluation of the variance with the time-frequency approach, is that we are integrating a smooth function, that is, the Wigner spectrum. Also, time-frequency distributions are very efficient in concentrating the information of signals in well localized regions of the time-frequency domain. The combination of smoothness and the concentration property is highly advantageous. The variance, $\sigma_{\boldsymbol{v}}^2(t)$, of a the process $\boldsymbol{v}(t)$, is

$$\sigma_{\boldsymbol{v}}^2(t) = E\left[\boldsymbol{v}^2(t)\right] \tag{23}$$

where we have assumed that $E\left[\boldsymbol{v}(t)\right] = 0$ for all times. We now show that indeed one can use the Wigner distribution to calculate it. In particular

$$\int \overline{W}_{\boldsymbol{v}}(t,\omega) d\omega = E\left[\boldsymbol{v}^2(t)\right] \tag{24}$$

and hence,

$$\sigma_{\boldsymbol{v}}^2(t) = \int \overline{W}_{\boldsymbol{v}}(t,\omega) d\omega \tag{25}$$

To illustrate we consider the specific example

$$\frac{d\boldsymbol{v}(t)}{dt} + \beta \boldsymbol{v}(t) = \boldsymbol{f}(t) \tag{26}$$

We have previously obtained the following exact solution

$$\overline{W}_{\boldsymbol{v}}(t,\omega) = -\frac{1}{\pi}\frac{N_0}{2\beta} e^{-2\beta t} \frac{\sin 2\omega t}{\omega} + \frac{N_0}{2\pi}\frac{1}{\beta^2+\omega^2} - \frac{N_0}{2\pi}\frac{e^{-2\beta t}}{\beta^2+\omega^2}\left(\cos 2\omega t - \omega/\beta \sin 2\omega t\right), \qquad t \geq 0 \tag{27}$$

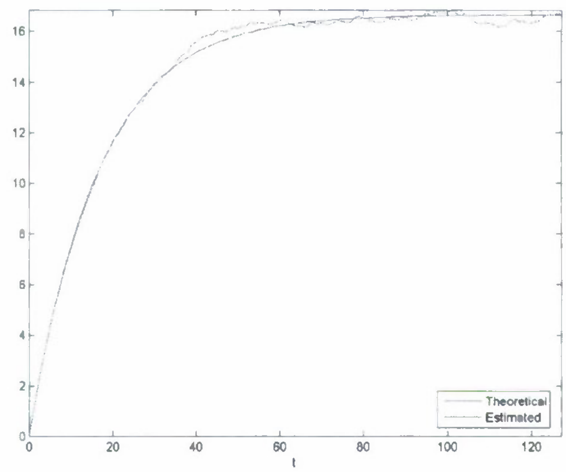

Figure 4: Estimated and theoretical variance of $v(t)$ of Brownian motion.

and using this one gets

$$\sigma_v^2(t) = \int \left[-\frac{1}{\pi}\frac{N_0}{2\beta} e^{-2\beta t} \frac{\sin 2\omega t}{\omega} + \frac{N_0}{2\pi}\frac{1}{\beta^2+\omega^2} - \frac{N_0}{2\pi}\frac{e^{-2\beta t}}{\beta^2+\omega^2} \left(\cos 2\omega t - \omega/\beta \sin 2\omega t\right) \right] d\omega \tag{28}$$

which simplifies to

$$\sigma_v^2(t) = \frac{N_0}{2\beta}\left[1 - e^{-2\beta t}\right], \qquad t \geq 0 \tag{29}$$

As an numerical example to confirm that the instantaneous variance $\sigma_v^2(t)$ can be obtained from the Wigner spectrum we performed a numerical simulation. We have generated the random differential equation with 10000 realizations and computed the variance using the Wigner spectrum. In Fig. 6 we show the estimated variance and the theoretical one. The agreement is excellent.

2.4 Transients

If a time-invariant system, possibly random, is turned on, it will have transients until it achieves steady-state. Often, the transients are very important. We have been able to prove two important properties regarding such systems:

a) We have shown that if we turn on the system at any finite time t_0, then the spectrum at infinity $t = \infty$ equals the standard power spectrum that is obtained with standard techniques.

b) Furthermore, we have been able to show that the time-frequency output can always be written as the sum of a transient part and a stationary part which corresponds to the power spectrum.

In general the equation for the Wigner spectrum can be written as

$$\left[b_{2n}\frac{\partial^{2n}}{\partial t^{2n}}+b_{2n-1}\frac{\partial^{2n-1}}{\partial t^{2n-1}}+\ldots+b_1\frac{\partial}{\partial t}+b_0\right]\overline{W}_{\mathbf{x}}(t,\omega)=u(t)\frac{N_0}{2\pi} \tag{30}$$

where $u(t)$ is the step function. We take the Laplace transform of Eq. (30) to obtain

$$\left[b_{2n}s^{2n}+b_{2n-1}s^{2n-1}+\ldots+b_1s+b_0\right]\overline{W}_{\mathbf{x}}(s,\omega)=\frac{N_0}{2\pi}\frac{1}{s} \tag{31}$$

Now, it follows that

$$S_\infty(\omega)=\lim_{t\to\infty}\overline{W}_{\mathbf{x}}(t,\omega)=\lim_{s\to 0}s\overline{W}_{\mathbf{x}}(s,\omega)=\frac{N_0}{2\pi}\frac{1}{b_0}=\frac{N_0}{2\pi}\left|H(\omega)\right|^2 \tag{32}$$

where $H(\omega)$ is the transfer function. This shows that when $t\to\infty$ the Wigner spectrum approaches the stationary solution. Now, it is always possible to write the Wigner spectrum $\overline{W}_{\mathbf{x}}(t,\omega)$ as a sum of a stationary part, constant with time, plus a transient spectrum. We write

$$\overline{W}_{\mathbf{x}}(t,\omega)=\overline{W}_S(t,\omega)+\overline{W}_T(t,\omega) \tag{33}$$

To show this fact, one decomposes $\overline{W}_{\mathbf{x}}(s,\omega)$ into a sum of partial fractions

$$\frac{\frac{N_0}{2\pi}}{s\left(b_{2n}s^{2n}+b_{2n-1}s^{2n-1}+\ldots+b_1s+b_0\right)}=\frac{N_0}{2\pi}\frac{A}{s}+\frac{\frac{N_0}{2\pi}N(s)}{b_{2n}s^{2n}+b_{2n-1}s^{2n-1}+\ldots+b_1s+b_0} \tag{34}$$

where A $(=\frac{1}{b_0}=|H(\omega)|^2)$ is a constant, and $N(s)$ is a polynomial in the complex variable s. One then may prove that the stationary and transient part of the Wigner spectrum are

$$\overline{W}_S(s,\omega)=\frac{N_0}{2\pi}\frac{A}{s}=\frac{N_0}{2\pi}\left|H(\omega)\right|^2\frac{1}{s} \tag{35}$$

$$\overline{W}_S(t,\omega)=\frac{N_0}{2\pi}u(t)\left|H(\omega)\right|^2 \tag{36}$$

$$\overline{W}_T(s,\omega)=\frac{\frac{N_0}{2\pi}N(s)}{b_{2n}s^{2n}+b_{2n-1}s^{2n-1}+\ldots+b_1s+b_0} \tag{37}$$

In the time domain,

$$\overline{W}_S(t,\omega)=\frac{N_0}{2\pi}u(t)\left|H(\omega)\right|^2 \tag{38}$$

which is exactly the stationary spectrum.

2.5 Extension to MIMO

Very often systems have multiple inputs and multiple outputs (MIMO systems) that are governed by differential equations of the form

$$\frac{d}{dt}\mathbf{X}(t)=M(t)\mathbf{X}(t)+\mathbf{F}(t) \tag{39}$$

where now $\mathbf{X}(t)$ and $\mathbf{F}(t)$ are column vectors with n elements

$$\mathbf{X}(t) = \begin{pmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_n(t) \end{pmatrix}, \qquad \mathbf{F}(t) = \begin{pmatrix} \mathbf{f}_1(t) \\ \mathbf{f}_2(t) \\ \vdots \\ \mathbf{f}_n(t) \end{pmatrix} \tag{40}$$

and $M(t)$ is consequently an n-by-n square matrix of complex coefficients. The derivative in Eq. (39) implies that every element of $\mathbf{X}(t)$ has to be differentiated with respect to time. Now we define the autocorrelation matrix of the solution vector $\mathbf{X}(t)$ as

$$R_{\mathbf{X}}(t_1, t_2) = E\left[\mathbf{X}(t_1)\mathbf{X}(t_2)^{\dagger}\right] \tag{41}$$

The autocorrelation of the input vector $\mathbf{F}(t)$ is defined in the identical way. We have shown that the equation for the autocorrelation of $\mathbf{X}(t)$ defined by Eq. (39) is

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}R_{\mathbf{X}} - \frac{\partial}{\partial t_1}R_{\mathbf{X}}M^{\dagger}(t_2) - \frac{\partial}{\partial t_2}M(t_1)R_{\mathbf{X}} + M(t_1)R_{\mathbf{X}}M^{\dagger}(t_2) = R_{\mathbf{F}} \tag{42}$$

where we have written $R_{\mathbf{X}}$, $R_{\mathbf{F}}$ instead of $R_{\mathbf{X}}(t_1, t_2)$, $R_{\mathbf{F}}(t_1, t_2)$ to keep the notation short.

2.6 Input-output relations for partial differential equations

We have developed input-output equations for systems governed by partial differential equations. We do not give here the equation, they can be found in our papers. We argue here that our method gives insight into the fundamental nature of certain partial differential equations. Consider for example two famous differential equations, the Schrodinger free particle equation and the heat equation.

$$\frac{\partial \psi}{\partial t} = ia\frac{\partial^2 \psi}{\partial x^2} \qquad \text{Shrodinger free particle; } a = \hbar/(2m) \tag{43}$$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \qquad \text{Diffusion equation; } D = \text{diffusion coefficient} \tag{44}$$

Much has been written as to the analogy of the two equations, and on the fact that one can be made equivalent to the other by considering an imaginary diffusion coefficient. We have transformed the equations into phase space where both become real. In this case the phase space is position and k space. The results are, respectively

$$\frac{\partial W_{\psi}}{\partial t} = -2ka\frac{\partial^2 W_{\psi}}{\partial x^2} \tag{45}$$

$$\frac{\partial W_u}{\partial t} = \frac{D}{2}\frac{\partial^2 W_u}{\partial x^2} - 2Dk^2 W_u \tag{46}$$

where

$$W_u(x,k;t) = \frac{1}{2\pi}\int u^*\left(x-\frac{\lambda}{2},t\right) u\left(x+\frac{\lambda}{2},t\right) e^{-jk\lambda}\, d\lambda \tag{47}$$

While the only difference between the two original equations, Eq. (43) and (44), is an i, the difference in phase space can be readily seen and studied, and indeed each of the terms has a physical interpretation. We have transformed other partial differential equations, and in each case the physical nature of the original equations is readily seen. Moreover, we have shown that solving partial differential equations in phase space is simpler.

2.7 A non-white noise model

We have developed an example for the non-white noise case that can be done analytically and can be used as a model example. Consider the random differential equation

$$\frac{d^2x(t)}{dt^2} + \beta\frac{dx(t)}{dt} = \xi(t) \tag{48}$$

where $\xi(t)$ is the noise operator that satisfies

$$\langle[\xi(t),\xi(t')]_+\rangle = \frac{\gamma\hbar}{\pi}\int_{-\infty}^{\infty} \omega e^{i\omega(t-t')} \coth\frac{\hbar\omega}{2kT} d\omega \tag{49}$$

It is called the *quasi-classical* Langevin equation and is standard in quantum optics. In such a case the autocorrelation function is

$$R(\tau) = \frac{\gamma\hbar}{2\pi}\int_{-\infty}^{\infty} \omega e^{i\omega\tau} \coth\frac{\hbar\omega}{2kT} d\omega \tag{50}$$

We rewrite it as

$$\dot{p}(t) + \beta p(t) = \xi(t) \tag{51}$$

with

$$R_\xi(\tau) = 2DZ\int_{-\infty}^{\infty} \omega e^{i\omega\tau} \coth Z\omega d\omega \quad ; \quad Z = \frac{\hbar}{2kT} \tag{52}$$

Using the standard Wiener-Khinchen theorem the power spectrum is given by

$$S_\xi(\omega) = 2DZ\omega \coth Z\omega \tag{53}$$

We note that as $Z \to 0$

$$\lim_{Z\to 0} R_\xi(\tau) = 2D\delta(\tau) \tag{54}$$

which is the white noise case. Using our general approach we have that the differential equation for the Wigner spectrum is

$$\left[\frac{1}{4}\frac{\partial^2}{\partial t^2} + \beta\frac{\partial}{\partial t} + \beta^2 + \omega^2\right]\overline{W}_p(t,\omega) = \frac{DZ}{\pi}\omega\coth Z\omega \tag{55}$$

and we have found the exact solution

$$\overline{W}_p(t,\omega) = \frac{1}{\beta^2+\omega^2}\frac{DZ}{\pi}\omega\coth Z\omega\left[1-e^{-\frac{2\gamma}{m}t}\cos 2\omega t\right] \tag{56}$$

2.8 Dispersive channels with damping

Channels are generally dispersive and also have damping. The fundamental nature of dispersive wave propagation is that different frequencies travel at different velocities, and hence one would expect that a transformation into a joint position-wavenumber representation would be well suited to study such dispersive propagation. Using our method we have shown that a pulse can be propagated in a very simple way without solving the governing differential equation. The classical method for solving a wave equation with constant coefficients,

$$\sum_{n=0}^{N} a_n \frac{\partial^n u}{\partial t^n} = \sum_{m=0}^{M} b_m \frac{\partial^m u}{\partial x^m} \tag{57}$$

is to substitute $e^{ikx-i\omega t}$ into it, to obtain a relation between k and ω,

$$\sum_{n=0}^{N} a_n(-i\omega)^n = \sum_{m=0}^{M} b_m(ik)^m \tag{58}$$

One then solves for ω in terms of k to obtain the dispersion relation $\omega = \omega(k)$. There may be more than one solution and each solution is called a mode. The general solution for a particular mode is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,0)\, e^{ikx-i\omega(k)t}\, dk \tag{59}$$

with

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\, e^{-ikx}\, dx \tag{60}$$

If one defines the time dependent spectrum by

$$S(k,t) = S(k,0)\, e^{-i\omega(k)t} \tag{61}$$

then $u(x,t)$ and $S(k,t)$ form Fourier transform pairs between x and k

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t)\, e^{ikx}\, dk \tag{62}$$

$$S(k,t) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-ikx}\, dx \tag{63}$$

For a complex dispersion relation

$$\omega(k) = \omega_R(k) + i\omega_I(k) \tag{64}$$

the group velocity, $v_g(k)$, is defined by

$$v_g(k) = \omega_R'(k) \tag{65}$$

The position-wave number Wigner distribution at time t is defined as

$$W(x,k,t)=\frac{1}{2\pi}\int u^*(x-\tfrac{1}{2}\tau,t)\,u(x+\tfrac{1}{2}\tau,t)\,e^{-i\tau k}\,d\tau \tag{66}$$

where $u(x,0)$ is the input. We obtained an approximation to $W(x,k,t)$ in terms of $W(x,k,0)$

$$W_a(x,k,t)=e^{2\omega_I(k)t}\,W(x-v_g(k)t,k,0)$$

where W_a is the approximate Wigner distribution. This is a remarkably simple result and besides being easily applied, this approximation is interesting and insightful because it is physically interpretable. In particular, each local point in phase space moves at constant velocity given by the group velocity, with attenuation governed by $\omega_I(k)$. We have tested this result on many exactly solvable examples, and it works much better than current approximation methods, such as the stationary phase approximation.

Ambiguity function. Extending this approximation approach to other representations is potentially useful. The reason is that while different representations have properties in common, sometimes different representations are more tractable. Of particular interest is the ambiguity function domain

$$M(\theta,\tau,t)=\int u^*(x-\tfrac{1}{2}\tau,t)\,u(x+\tfrac{1}{2}\tau,t)\,e^{i\theta x}dx \tag{67}$$

We note that the ambiguity function and the Wigner distribution are a Fourier transform pair. We have been able to show that

$$M(\theta,\tau,t)=\frac{1}{2\pi}\int\int M(\theta,\tau',0)e^{ik(\tau-\tau')}\;e^{it[\omega^*(k+\frac{1}{2}\theta)-\omega(k-\frac{1}{2}\theta)]}\;d\tau'\,dk \tag{68}$$

This is exact. We have also been able to show that the approximate ambiguity function is

$$M_a(\theta,\tau,t)=\frac{1}{2\pi}\int\int e^{2\omega_I(k)t}\,M(\theta,\tau',0)\,e^{ik(\tau-\tau')}\,e^{itv_g(k)\theta}\;d\tau'\,dk \tag{69}$$

Spectrogram. The spectrogram is perhaps the most widely used distribution and goes under many names, for example in acoustics it is called the "lofargram". It is often used to refer to the time versus frequency plot of signal energy obtained by a filter-bank analysis. Since we are considering the phase space of position and wave number, we define the "local" (or "short-space") wave number transform by

$$S_x(k,t)=\frac{1}{\sqrt{2\pi}}\int\;u(x',t)h^*(x-x')e^{-ikx'}\;dx' \tag{70}$$

where $h(x)$ is a spatial window function that is typically narrower in extent than the wave $u(x,t)$, peaks at $x=0$ and tapers to zero for $|x|>0$, although this is not required in what follows. The position-wave number spectrogram is then given by

$$P_{sp}(x,k,t,)=|\,S_x(k,t)\,|^2\;=\left|\frac{1}{\sqrt{2\pi}}u(x',t)h^*(x-x')e^{-ikx'}dx'\right|^2 \tag{71}$$

We have found the following approximate solution

$$P_{sp}(x,k,t) \approx \frac{1}{2\pi} \int e^{2\omega_I((k'+k'')/2)t}\, S(k',0)S^*(k'',0)\; H^*(k-k')$$
$$H(k-k'')\; e^{-i(x-v_g((k'+k'')/2)t)(k''-k')}\; dk'dk'' \tag{72}$$

Notice now that this no longer can be put in the spectrogram form, that is, as a magnitude-square of some integral function.

2.9 Position-Wavenumber Approximation for Random Dispersive Channels

We derived an approximation of the position-wavenumber (x-k) Wigner spectrum of the random channel, and its impact on a propagating wave. When there is no dispersion and the channel is deterministic, the pulse does not spread as it propagates. However, in a random channel, the pulse spreads in the mean even in the absence of dispersion. Let $u(x,0)$ be an initial deterministic pulse, propagating in a random channel with Wigner spectrum $\overline{W}_h(x,k;\,t)$. The propagating wave $u(x,t)$ is therefore also random, and its Wigner spectrum $\overline{W}_u(x,k;\,t)$ is given by [1, 7]

$$\overline{W}_u(x,k;\,t) \;=\; E\left\{\frac{1}{2\pi}\int u^*\left(x-\frac{\lambda}{2},t\right) u\left(x+\frac{\lambda}{2},t\right)\, e^{-jk\lambda}\, d\lambda\right\} \tag{73}$$

$$= \int W_u(x-x',k;\,0)\,\overline{W}_h(x',k;\,t)\,dx' \tag{74}$$

where $W_u(x,k;\,0)$ is the Wigner distribution of $u(x,0)$ and $E\{.\}$ denotes the expected value. The approach is to extend the deterministic case to the random case by introducing random parameters, and then ensemble averaging the deterministic approximation.

As a case/example, we consider a dispersive channel with random speed c and exponential attenuation parameterized by the variable b, with joint probability distribution $P(b,c)$. For this case, the approximate Wigner spectrum of the channel is given by,

$$\overline{W}_h(x,k;\,t) \approx E\left\{e^{-bkt}\,\delta(x-v_g(k)\,t)\right\} \;=\; \frac{1}{|t\,v_1(k)|}\int e^{-bkt}\,P\left(b,\frac{x}{t\,v_1(k)}\right)\,db$$

where $v_g(k) = c\,v_1(k)$ is the group velocity, parameterized by the random sound speed c and the deterministic function $v_1(k)$. The dispersionless case corresponds to $v_1(k) = 1$.

Let the joint probability distribution function $P(b,c)$ be independent, with sound speed c described by arbitrary distribution $P_c(c)$ and the damping parameter b given by an Erlang distribution,

$$P_b(b) = \frac{1}{(n-1)!}\,\lambda^n\, b^{n-1} e^{-\lambda b}, \quad b \geq 0 \tag{75}$$

The Wigner spectrum of the channel is then approximately

$$\overline{W}_h(x,k;\,t) \approx \left(\frac{\lambda}{\lambda+kt}\right)^n \frac{1}{|t\,v_1(k)|}\,P_c\left(\frac{x}{t\,v_1(k)}\right) \tag{76}$$

To obtain the approximate Wigner spectrum of the propagating wave, we convolved this channel spectrum with the Wigner spectrum of the initial wave. Note that as the wave evolves the channel increasingly attenuates the propagating wave, and the statistics of the wave in x are distributed according to $P_c\left(\frac{x}{t\, v_1(k)}\right)$. If we take $P_c(e)$ to be Gaussian with mean c_0 and variance σ^2, then the approximation is

$$\overline{W}_h(x,k;\,t) \approx \left(\frac{\lambda}{\lambda + kt}\right)^n \frac{1}{\sqrt{2\pi\, t^2\, v_1^2(k)\,\sigma^2}} e^{-\frac{(x - c_0\, v_1(k)\, t)^2}{2\, t^2\, v_1^2(k)\,\sigma^2}} \tag{77}$$

Note that at $t = 0$ the wave has not propagated, and the channel approximation becomes $\delta(x)$, which is a satisfying result in that this result is exactly satisfied by the approximation at $t = 0$. This is not true of other standard approximations, such as the stationary phase approximation, which is not accurate for small times. Also, note that even when there is no dispersion ($v_1(k) = 1$), the channel still induces spreading on average in the wave, unlike the deterministic case for which the channel Wigner spectrum becomes $\delta(x - ct)$ in the absence of dispersion (and damping).

2.10 Invariant Features for Classification

As a pulse or wave $u(x,t)$ propagates, it can change its shape due to factors such as frequency-dependent attenuation and dispersion by the propagation channel. This can have a deleterious impact on detection and classification. Hence one asks is there are features that are invariant to propagation effects. We have derived new central-like moments that are invariant to dispersive propagation as we know explain.

Let $u(0,t)$ denote the initial pulse as a function of time t, generated at position $x = 0$. Then, the pulse at a subsequent position is given by [4, 5, 9, 10]

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int F(0,\omega)\, e^{jK(\omega)x}\, e^{-j\omega t}\, d\omega \;=\; \frac{1}{\sqrt{2\pi}} \int F(x,\omega)\, e^{-j\omega t}\, d\omega \tag{78}$$

per mode, where $F(0,\omega)$ is the spectrum of the initial pulse,

$$F(0,\omega) = \frac{1}{\sqrt{2\pi}} \int u(0,t)\; e^{j\omega t}\, dt \tag{79}$$

$K(\omega)$ is the dispersion relation expressed in terms of k as a function of ω. If the dispersion relation is complex, then there is damping, resulting in frequency-dependent attenuation as the wave propagates. Also, we define the amplitude and phase of the wave and its Fourier transform by

$$u(x,t) = A(x,t) e^{j\varphi(x,t)} \tag{80}$$

$$F(x,\omega) \;=\; B(x,\omega)\, e^{j\psi(x,\omega)} \tag{81}$$

Note that

$$F(0,\omega) \;= B(0,\omega)\, e^{j\psi(0,\omega)} \tag{82}$$

and therefore

$$F(x,\omega) = F(0,\omega)\, e^{jK(\omega)x} \tag{83}$$

$$= B(0,\omega)\, e^{j\psi(0,\omega)}\, e^{jK(\omega)x} \tag{84}$$

It follows that, for the real dispersion relation $K(\omega)$,

$$B(x,\omega) = B(0,\omega) \tag{85}$$

$$\psi(x,\omega) = \psi(0,\omega) + K(\omega)x \tag{86}$$

The central-like moments that are invariant to dispersive propagation are given by

$$A_n(x) = \int F^*(x,\omega) \left(j\frac{\partial}{\partial\omega} - t_g(x,\omega) \right)^n F(x,\omega)d\omega \tag{87}$$

where $t_g(x,\omega) = -\psi'(x,\omega)$ is the *group delay* of the wave. These moments are similar to central temporal moments, but with an important difference. To see the similarity, consider ordinary central temporal moments of order n, given by

$$\langle g(t)\rangle_x = \int u^*(x,t)\,(t - \langle t\rangle_x)^n\, u(x,t)\,dt \tag{88}$$

$$= \int F^*(x,\omega) \left(j\frac{\partial}{\partial\omega} - \langle t\rangle_x \right)^n F(x,\omega)d\omega \tag{89}$$

where Eq. (89) follows from Eq. (88) by substituting in for $u(x,t)$ in terms of its Fourier transform, and $\langle t\rangle$ is the mean time. (Note that for $n = 2$ this is the duration moment.) We see that the moments $A_n(x)$ are similar to ordinary central temporal moments, except that rather than being centered about the average time $\langle t\rangle_x$, they are centered about the group delay $t_g(x,\omega)$ of the wave.

This centering about the group delay affords these moments with a property that is particularly attractive to their use as potential features for classification, namely, they do not change with propagation, even though the wave $u(x,t)$ does. That is, they are independent of location x, [2, 6],

$$A_n(x) = A_n(0) \tag{90}$$

for real dispersion relations. Thus, if differences are observed in these moments for two different received pulses, the differences reflect differences in the source rather than differences due to propagation (i.e. dispersion) effects. The moments $A_n(x)$ can be equivalently calculated in the time domain [2, 6], which is beneficial for computational purposes. In particular, the equivalent time-domain formulation is

$$A_n(x) = \frac{1}{\sqrt{2\pi}} \int\int t^n\, |F(x,\omega)|\; e^{-j\omega t} d\omega\, dt \tag{91}$$

Also, as defined, the odd-order moments are identically zero. Hence we use a one-sided integral over t in Eq. (91) to obtain non-zero odd-order dispersion-invariant moments.

It is important to appreciate that the results above are for a mode. Since in general each mode travels with a different group velocity (i.e., each mode has a different dispersion relation), the spectral and A_n moments of the total wave, which is comprised of the sum of modes, change with distance, even though the spectral and A_n moments of each mode are invariant to dispersive propagation. Thus, mode separation prior to calculating the moments is important in order to obtain invariant moments. However, even without mode separation, the moments A_n will be less variable with propagation distance than will ordinary temporal moments, and may therefore still be better features for classification.

We have compared classification performance to that of ordinary temporal moments on numerical models of acoustic scattering from steel shells in a Pekeris waveguide. We evaluated the two-class problem of distinguishing a sphere from a cylinder, and the more challenging problem of distinguishing between two different cylinders, via their backscattered echoes at various propagation distances. The classification utility of our moment features was assessed by computing receiver operating characteristic (ROC) curves for each two-class problem. We also considered an approach based on a correlator receiver. We applied our dispersion-invariant approach to define a dispersion-invariant correlation coefficient for use as a classification feature, and have shown the detrimental effects of dispersion on the ordinary correlation coefficient. We calculated ROC curves and have shown the significant performance gains over the ordinary correlation coefficient for the two-class simulations.

2.11 Fields

We developed a direct method using phase space distributions, such as the Wigner distribution, to study wave propagation in a dispersive medium. We derived an explicit evolution equation for the Wigner distribution and methods to solve it. This has been described in the previous sections. We have also developed the method to the evolution of noise fields in a dispersive medium. For linear wave equations with constant coefficients the solution is given by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,0)\, e^{-i\omega(k)t}\, e^{ikx}\, dk \tag{92}$$

where $S(k,0)$ is the initial spatial spectrum which is obtained from the initial pulse by way of

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\, e^{-ikx}\, dx \tag{93}$$

If we define

$$S(k,t) = S(k,0)\, e^{-i\omega(k)t} \tag{94}$$

in which case

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t)\, e^{ikx}\, dk \tag{95}$$

$$S(k,t) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-ikx}\, dx \tag{96}$$

Differentiating Eq. (92) with respect to time and multiplying by i, gives

$$i\frac{\partial}{\partial t}u(x,t) = \frac{1}{\sqrt{2\pi}} \int \omega(k) S(k,0)\, e^{-i\omega(k)t}\, e^{ikx}\, dk \tag{97}$$

Now, in general, the Fourier transform of the product of two functions can simplified by the use of the following relation,

$$\int f(k) g(k) e^{ikx}\, dk = \int f\left(\frac{1}{i}\frac{\partial}{\partial x}\right) g(k) e^{ikx}\, dk = f\left(\frac{1}{i}\frac{\partial}{\partial x}\right) \int g(k) e^{ikx}\, dk \tag{98}$$

Applying this to Eq. (97) we have that

$$i\frac{\partial}{\partial t}u(x,t) = \omega\left(\frac{1}{i}\frac{\partial}{\partial x}\right) u(x,t) \tag{99}$$

We rewrite this as

$$i\frac{\partial}{\partial t}u(x,t) = \omega(\mathcal{K}) u(x,t) \tag{100}$$

where $\mathcal{K}$ is the "wave number operator" defined by

$$\mathcal{K} = \frac{1}{i}\frac{\partial}{\partial x} \tag{101}$$

We derived the equation of motion for the Wigner distribution

$$i\frac{\partial}{\partial t}W(x,k,t) = \left[\omega\left(k + \frac{1}{2i}\frac{\partial}{\partial x}\right) - \omega\left(k - \frac{1}{2i}\frac{\partial}{\partial x}\right)\right] W(x,k,t) \tag{102}$$

We note that this equation is derived under the assumption that ω is real. In Eq. (102) what is meant by $\omega\big(k + \frac{1}{2i}\frac{\partial}{\partial x}\big)$ is that we substitute the operator $k + \frac{1}{2i}\frac{\partial}{\partial x}$ in $\omega(k)$.

Noise fields: The problem we have done is the following. We consider the case where at a spatial point, x_0, a noise field is produced as a function of time and we ask what the field is at a later time. One has that

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int F(\omega)\, e^{iK(\omega)(x-x_0)-i\omega t} d\omega \tag{103}$$

where $K(\omega)$ is the dispersion relation as a function of ω and where $F(\omega)$ is the initial time spectrum at $x = x_0$,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int u(x_0,t)\, e^{i\omega t} dt \tag{104}$$

By defining

$$F(\omega, x) = F(\omega, x_0)\, e^{iK(\omega)x} \tag{105}$$

where

$$F(\omega, x_0) = F(\omega) \tag{106}$$

we have that

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int F(\omega, x)\, e^{-i\omega t} d\omega \tag{107}$$

$$F(\omega, x) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{i\omega t} dt \tag{108}$$

In addition it is convenient to combine Eqs. (103) and (104)

$$u(x,t) = \frac{1}{2\pi} \iint u(x_0, t')\, e^{iK(\omega)(x-x_0)-i\omega t} e^{i\omega t'} dt' d\omega \tag{109}$$

Let us now study the statistical properties of $u(x,t)$ considered as a random field. We first ask the following question. Suppose the noise field is stationary at position of generate x_0, will it be time invariant at other positions? In particular consider

$$u^*(x,t)\, u(x,t+\tau) = \left(\frac{1}{2\pi}\right)^2 \iint u^*(x_0,t') u(x_0,t'')\, e^{-iK(\omega)(x-x_0)+i\omega t} e^{-i\omega t'} e^{iK(\omega')(x-x_0)-i\omega'(t+\tau)} e^{i\omega' t''} dt' d\omega dt'' d\omega' \tag{110}$$

and take the ensemble average

$$E[u^*(x,t)\, u(x,t+\tau] = \left(\frac{1}{2\pi}\right)^2 \iint E[u^*(x_0,t') u(x_0,t'')]\, e^{-iK(\omega)(x-x_0)+i\omega t} e^{-i\omega t'} e^{iK(\omega')(x-x_0)-i\omega'(t+\tau)} e^{i\omega' t''} dt' d\omega dt'' d\omega' \tag{111}$$

where we have assumed that $K(\omega)$ is a deterministic function. Suppose we define R by

$$R(x_0, t''-t') = E[u^*(x_0,t') u(x_0,t'')] \tag{112}$$

where we have explicitly assumed stationarity. Hence

$$E[u^*(x,t)\, u(x,t+\tau)] = \left(\frac{1}{2\pi}\right)^2 \iint R(x_0, t''-t')\, e^{-iK(\omega)(x-x_0)+i\omega t} e^{-i\omega t'} e^{iK(\omega')(x-x_0)-i\omega'(t+\tau)} e^{i\omega' t''} dt' d\omega dt'' d\omega' \tag{113}$$

We have been able to show that this reduces to $R(x_0, t'')$. Thus we see that if the process is stationary at position x_0 it will be stationary at any point in space. We now consider the same

issue from a different perspective. We define the Wigner distribution in t, ω of $u(x,t)$ at a particular position x by:

$$W(t,\omega;\, x) = \frac{1}{2\pi}\int u^*(x, t-\tau/2)\, u(x, t+\tau/2)\, e^{-i\omega\tau}d\tau \tag{114}$$

One defines the Wigner spectrum by

$$\overline{W}(t,\omega;x) = \frac{1}{2\pi}\int E[u^*(x,t-\tau/2)u(x,t+\tau/2)]e^{-i\tau\omega}d\tau \tag{115}$$

We have shown that

$$W(t,\omega;\, x) = \frac{1}{2\pi}\int\int W(t',\omega;x_0)\, e^{-i\theta(t'-t)}e^{-i(x-x_0)[k^*(\omega+\theta/2)-k(\omega-\theta/2)]}dt'd\theta \tag{116}$$

which relates the Wigner distribution at position x to the Wigner distribution of the initial pulse at $x = x_0$. We now take the ensemble average of both sides of Eq. (116).

$$\overline{W}(t,\omega;\, x) = \frac{1}{2\pi}\int\int \overline{W}(t',\omega;\, x_0)\, e^{-i\theta(t'-t)}e^{-i(x-x_0)[k^*(\omega+\theta/2)-k(\omega-\theta/2)]}dt'd\theta \tag{117}$$

Now, let us assume that $\overline{W}(t',\omega;\, x_0)$ is independent of time which is the condition of a stationary process. Let us call it $\overline{W}(\omega;\, x_0)$. We have

$$\overline{W}(t,\omega;\, x) = \int\int \overline{W}(\omega;\, 0)\, \delta(\theta)e^{-i\theta(t'-t)}e^{-ix[k^*(\omega+\theta/2)-k(\omega-\theta/2)]}dt'd\theta = \overline{W}(\omega;\, 0) \tag{118}$$

which shows that the Wigner distribution is also stationary at position x which is consistent with what we derived above.

In addition we consider the same problem using the approximate Wigner distribution. We have shown that an very accurate approximation can be obtained by way of

$$W(t,\omega;\, x) \approx W(t-\tau_g(\omega)(x-x_0),\omega;x_0) \tag{119}$$

where $\tau_g(\omega)$ is the derivative of the dispersion

$$\tau_g(\omega) = k'(\omega) = \frac{dk(\omega)}{d\omega} \tag{120}$$

and called the "group slowness", or "unit transit time".

Now, take the ensemble average of Eq. (118)

$$\overline{W}(t,\omega;x) \approx \overline{W}(t-\tau_g(\omega)(x-x_0),\omega;x_0) \tag{121}$$

and it is clear that the approximation also satisfies Eq. (118). We emphasize that the above considerations is for the case where the dispersion relation is deterministic. In many cases of course that is not the case, and such situation is under development.

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